It is to be recalled that the denominator of (33) is formed of radical factors of those roots of $L(p)$, p_j, which correspond to a negative real part of $p_j/\sqrt{1+p_j^2}$. Since these roots are, in turn, the poles of $k(p)$ which remain fixed irrespective of the choice of the roots of $k(p)$, (33) is invariant to all structures having the same insertion loss function.[4]

We next show the terminating transformer invariance. With respect to Fig. 1 we find that a specification of a terminating transformer in a basic pattern implies the existence of the inverse transformer as well. The terminating transformer is found from the insertion loss for $\theta=0$ and is given through the relationship

$$4(L(0)-1)=\left(N-\frac{1}{N}\right)^2 \quad (34)$$

[4] Equation (33) is tantamount to a minimum phase statement. Since $L(p)\to\infty$ as $p\to\infty$, the transmission function $t(p)$ vanishes for $\zeta=1$ in the ζ plane. One cannot, therefore, make any direct minimum phase statements because of the nonanalyticity of ln (t) in the right-half ζ plane.

so that the transformer is specified to within an inverse. Since the insertion loss is the invariant specification to all the multiple syntheses, the transformer is an invariant to the basic root pattern. One may, therefore, always construct at least one quarter-wave transformer working into the same real impedance N^2 for each basic pattern.

References

[1] Riblet, H. J., General synthesis of quarter-wave impedance transformers, *IRE Trans. on Microwave Theory and Techniques*, vol MTT-5, Jan 1957, pp 36–43.

[2] Seidel, H., Synthesis of a class of microwave filters, *IRE Trans. on Microwave Theory and Techniques*, vol MTT-5, Apr 1957, pp 107–114.

[3] Feldshtein, A. L., Synthesis of stepped directional couplers, *Radiotekhnika i Elektronika*, vol 6, Feb 1961, pp 234–240.

[4] Young, L., *Proc. IEE*, vol 110, pt B, Feb 1963, pp 275–281.

[5] Levy, R., General synthesis of asymmetric multi-element coupled-transmission-line directional couplers, *IEEE Trans. on Microwave Theory and Techniques*, vol MTT-11, Jul 1963, pp 226–237.

[6] Darlington, S., Synthesis of reactance four-poles which produce prescribed insertion loss characteristics, *J. Math. Phys.*, vol 18, Sep 1939, pp 257–353.

[7] Richards, P. I., A special class of functions with positive real part in a half plane, *Duke Math. J.*, vol 14, Sep 1947, pp 777–786.

Direct Synthesis of Band-Pass Transmission Line Structures

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Abstract—**Realizable band-pass (zero of transmission, i.e., infinite loss, at dc) equiripple gain functions are constructed which permit exact physical realization of systems consisting of cascaded lines and stubs. The problem of the realization of a prescribed load resistance is solved when a dc zero of transmission is present due to a shunt short-circuiting stub. The exact limits of realizable load resistance are given for equiripple band-pass gain functions and a straightforward method is presented to synthesize any desired value of load between the predetermined limits. The basis of the synthesis technique is the choice of location of the shunt stub in the cascaded chain. It is shown that the load resistance decreases monotonically as the distance of the stub from the generator increases, and it is this property which permits the realization of a wide range of load resistance from a given gain function. The method is illustrated by designs of filters, as well as a new form of broadband transformer in which the low-frequency response is suppressed by shunt stubs.**

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I. Introduction

A. Application of Band-Pass Transmission Line Functions

The SYNTHESIS of cascaded, lossless, commensurate transmission line circuits is well established [1]–[11]. The results may be summarized by stating the necessary and sufficient conditions for the realizability of such a cascaded line structure [1]: Given a transmission scattering coefficient $s_{12}(j\beta l)$ [β is the propagation constant, l the line length] such that under the transformation

$$\Omega=\tan\beta l$$

the amplitude function $|s_{12}(j\Omega)|^2$ is rational, then for a cascade of n lines each of length l, $|s_{12}|^2$ must be of the form

$$|s_{12}(j\Omega)|^2 = \frac{(1+\Omega^2)^n}{P_n(\Omega^2)}$$

where P_n is an even polynomial of degree n in the real variable Ω^2 and

$$0 \le |s_{12}(j\Omega)|^2 \le 1, \qquad 0 \le \Omega^2 \le \infty.$$

Subject to these constraints it is then possible to choose the function $|s_{12}(j\Omega)|^2$ so as to approximate desirable low-pass filter characteristics. One can also represent the functional form for a broadband impedance transformer over a prescribed band, provided at dc $|s_{12}(j\Omega)|^2$ actually takes on the proper value associated with the mismatch of the load and generator resistances.

One limitation of the cascaded line function is the fact that at dc the system is completely transparent so that a band-pass response which provides zero transmission at low frequencies is not realizable. If a short-circuited stub line is connected in shunt with the cascade of transmission lines, the transmission of the system will go to zero at dc. Such a network can then exhibit band-pass filter characteristics and, hence, a consideration of functions which describe cascaded line networks and shunt stubs is indicated.

A further application of cascaded line-stub network functions is indicated when one considers a broadband transformer for a load consisting of a resistor shunted by an open or short-circuited transmission line. Some typical examples of such loads are a bolometer in waveguide backed by a quarter-wave short circuit, a coaxial to waveguide adapter which involves a probe extending from the coaxial line into the waveguide and backed in the guide by a short-circuited length of line, a tunnel diode termination which can be approximated by a negative resistor shunted by an open-circuited stub line, and a microwave absorber which can be represented by a resistor shunted by a short-circuited stub line.

In the development of cascaded line-stub network functions which follows, emphasis is placed on band-pass filter network functions and their synthesis, but the functions derived find application to the other devices mentioned previously.

B. Properties of Cascaded Line Networks with Shunt Stubs

The use of a transmission line function for cascaded lines without stubs has been very completely discussed by Ozaki and Ishii [5]. They consider scattering functions of the form

$$|s_{12}(j\Omega)|^2 = \frac{(1+\Omega^2)^n(\Omega^2-\Omega_1^2)^2\cdots(\Omega^2-\Omega_k^2)^2}{P_m(\Omega^2)},$$

$$m \ge n + 2k$$

and discuss means for realizing appropriate functions of this form by cascaded lines with shunting open-circuited stubs. Further, they discuss a conformal mapping technique which will allow the choice of P_m as well as the real frequency transmission zeros, $\Omega_1, \Omega_2, \cdots, \Omega_k$ so as to obtain equal-ripple response in pass and stop bands. It is to be noted that the $|s_{12}(j\Omega)|^2$ coefficient as previously given is essentially a low-pass function, and can be used only for a band-pass response by operating in one of the higher periods of the variable $\Omega = \tan \beta l$, i.e., $\pi < \beta l$. A band-pass response with a zero of transmission at dc requires a short-circuited stub and for this case a very simple technique based on Sharpe's work [12], [13]–[15] can be used to obtain equal-ripple response in a band-pass region. This method avoids any use of the potential analogy with subsequent charge quantization [8] and yields simple explicit response functions in terms of Chebyshev polynomials. It is possible to extend the technique to the multiple stub case also, utilizing higher order zeros of transmission at $\Omega = 0$ (dc).

An important result of Ozaki's work [5] is the statement of sufficient conditions on the location of the Ω_k, so that in the synthesis of low-pass functions only cascaded lines and open-circuited stubs are required, and no mutually coupled coils or transformers are needed. Furthermore, the low-pass function can always be arranged so that

$$|s_{12}(0)|^2 = \frac{4R_1R_G}{(R_1+R_G)^2}$$

and this assures one that a prescribed load resistance will be obtained in the synthesis. These results have not been extended to the exact synthesis of the general band-pass case with short-circuited and open-circuited stubs, nor has a criterion been established which enables one to predict in advance that the synthesis will terminate in a prescribed load, since the zero of transmission at dc obscures the input effect of the load resistance. However, in the case of cascaded transmission line functions with a simple zero of transmission at dc, as discussed subsequently, it is possible to describe an exact synthesis procedure, using at most, two short-circuited stubs, which provides a wide range of control over the terminating load resistance depending on where, along the cascaded line structure, the stubs are removed. The direct synthesis, without a low-pass to band-pass transformation, of band-pass transmission line filters for prescribed generator and load terminations as described here, in which infinite loss occurs at dc, has not been presented elsewhere in the technical literature to the authors' knowledge.

II. Development of Equal-Ripple Functions for Cascaded Lines and a Single Stub [14], [15]

The case of a cascade of n lossless transmission lines and a shunt short-circuited stub, all elements having equal length, will be initially considered. The stub introduces a simple zero of transmission at $\Omega=0$; therefore, for this network the insertion gain function which is equal to the squared amplitude of the transmission scattering coefficient $|s_{12}|^2$ has the generic form:

$$|s_{12}(j\Omega)|^2 = \frac{\Omega^2(1+\Omega^2)^n}{P_{n+1}(\Omega^2)} \tag{1}$$

with

$$0 \le |s_{12}(j\Omega)^2|^2 \le 1$$

where $\Omega=\tan\theta$, $\theta=\beta l$, and P_{n+1} is a polynomial of degree $(n+1)$ in Ω^2. The function s_{12} is a normalized scattering coefficient, with normalization numbers equal to the prescribed load and generator [16] terminations.

Therefore, substituting for Ω in the foregoing, one may alternately express the insertion gain as:

$$|s_{12}|^2 = \frac{\tan^2\theta(\sec^2\theta)^n}{P_{n+1}(\tan^2\theta)} = \frac{\sin^2\theta(\cos^2\theta)^{-(n+1)}}{P_{n+1}(\tan^2\theta)} = \frac{\sin^2\theta}{P_{n+1}(\cos^2\theta)}. \tag{2}$$

Now let $x\equiv\alpha\cos\theta$ for which $\sin^2\theta=1-\cos^2\theta=(\alpha^2-x^2)/\alpha^2$. Under this transformation,

$$|s_{12}|^2 = \frac{\alpha^2-x^2}{G_{n+1}(x^2)} = \frac{\alpha^2-x^2}{(\alpha^2-x^2)+H_{n+1}(x^2)} = \frac{1}{1+\dfrac{H_{n+1}(x^2)}{\alpha^2-x^2}} = \frac{1}{1+F(x^2)} \tag{3}$$

where

$$H_{n+1}(x^2) \equiv G_{n+1}(x^2) - (\alpha^2-x^2)$$

and

$$F(x^2) \equiv \frac{H_{n+1}(x^2)}{\alpha^2-x^2},$$

with G_{n+1}, and H_{n+1} polynomials of degree $(n+1)$ in x^2.

The basic problem of equal-ripple specification will be, therefore, to determine a functional form for $F(x^2)$ which will both exhibit equal-ripple pass band behavior and possess the requisite poles at $x=\pm\alpha$. To accomplish this, a modification of the basic Chebyshev polynomial generating function will be employed.

Let $x=\cos\phi=\alpha\cos\theta$; that is, $\phi=\cos^{-1}(\alpha\cos\theta)$. Furthermore, let $F(x^2)\equiv\cos(2n\phi+\delta)$, where the angle δ is to be specified.

The pass band is to comprise the interval $-1\le x\le 1$. As x varies from $+1$ to -1, the angle ϕ covers π radians, or $2n\phi$ covers $2n\pi$ radians. To achieve pass band equal ripple, the angle δ is to be constrained in such a fashion that it monotonically traverses 2π radians as x varies from $+1$ to -1. Under this constraint, the effect of the angle δ in the pass band will be to merely add an additional ripple though not of the same shape factor as that due to the variation of ϕ. If these properties, together with the essential constraint that $F(x^2)=\cos(2n\phi+\delta)$ be a rational function, can be satisfied, equal-ripple specification will be accomplished.

Consider the following function:

$$\cos\delta \equiv \frac{(2\alpha^2-1)x^2-\alpha^2}{\alpha^2-x^2}. \tag{4}$$

To insure the monotonic variation of the angle δ within the range $-1\le x\le 1$, the slope $d\delta/dx$ must not change sign. Evaluating this:

$$\frac{d\delta}{dx} = \frac{-4x\alpha^2(\alpha^2-1)}{(\alpha^2-x^2)^2\sin\delta}. \tag{5}$$

Within the range $0\le x\le +1$, δ is in the 1st and 2nd quadrants since within this range $\cos\delta$ goes from $+1$ through 0 to -1. Hence, $\sin\delta$ is positive and $(d\delta/dx)<0$. Within the range $-1\le x\le 0$, δ is in the 3rd and 4th quadrants. Since both x and $\sin\delta$ are negative in this region, $d\delta/dx$ is again <0. Therefore, the slope of δ vs. x is negative throughout the entire range; the variation is monotonic. Hence, the argument $(2n\phi+\delta)$ ranges over $(n+1)$ cycles of 2π across the pass band; $\cos(2n\phi+\delta)$ repeats $(n+1)$ times with $2(n+1)$ zeros.

The function $\cos(2n\phi+\delta)$ varies between ± 1 in the pass band, that is, $-1\le\cos(2n\phi+\delta)\le 1$, for $-1\le x\le 1$. Therefore, to insure the boundedness of $|s_{12}|^2$ by unity, the insertion gain function may be specified as:

$$|s_{12}|^2 = \frac{1-\epsilon^2}{1+\epsilon^2F(x^2)} = \frac{1-\epsilon^2}{1+\epsilon^2\cos(2n\phi+\delta)}. \tag{6}$$

For this specification, $0\le|s_{12}|^2\le 1$, since the denominator is always $\ge 1-\epsilon^2$. The function $\cos(2n\phi+\delta)$ is $>+1$ for $1<|x|<\alpha$.

The remaining property that must be considered is that of rationality. To demonstrate this, consider $\cos(2n\phi+\delta)$ in its expanded form:

$$\cos(2n\phi+\delta) = \cos 2n\phi\cos\delta - \sin 2n\phi\sin\delta.$$

The product $\cos 2n\phi\cos\delta$ is rational since $\cos 2n\phi = T_{2n}(x)$, the rational Chebyshev polynomial of the first kind, while $\cos\delta$ is, by definition, the ratio of rational functions previously stated. The function $\sin 2n\phi = U_{2n}(x)$, is a Chebyshev function of the second kind. This function is related to a rational function through the identity:

$$U_{2n}(x) = \sqrt{1-x^2}\,Q_{(2n-1)}(x) \text{ where}$$

$Q_{(2n-1)}(x)$ is a rational polynomial.

The remaining term is

$$\sin\delta = \sqrt{1-\cos^2\delta} = \frac{2\alpha x\sqrt{(1-x^2)(\alpha^2-1)}}{\alpha^2-x^2}$$

where we have chosen the positive sign for the square root to assure that $\sin\delta\ge 0$ for $x\le 1$, noting that $\alpha\ge 1$.

Therefore, $\sin 2n\phi \sin \delta$ is likewise rational. Thus,

$$\cos(2n\phi+\delta) = \frac{([2\alpha^2-1]x^2-\alpha^2)T_{2n}(x)-2\alpha x(1-x^2)\sqrt{\alpha^2-1}\,Q_{2n-1}(x)}{\alpha^2-x^2} \quad (7)$$

is rational and possesses the requisite poles at $x=\pm\alpha$.

An alternate specification for $|s_{12}|^2$ can be achieved by the definition of a new angle, $\xi=\delta/2$, for which:

$$\cos\xi = \sqrt{\frac{1+\cos\delta}{2}} = x\sqrt{\frac{\alpha^2-1}{\alpha^2-x^2}} \quad (8a)$$

$$\sin\xi = \sqrt{1-\cos^2\xi} = \alpha\sqrt{\frac{1-x^2}{\alpha^2-x^2}}. \quad (8b)$$

Thus:

$$\cos(n\phi+\xi) = \cos n\phi\cos\xi - \sin n\phi\sin\xi = T_n(x)\cdot x\sqrt{\frac{\alpha^2-1}{\alpha^2-x^2}} - U_n(x)\cdot\alpha\sqrt{\frac{1-x^2}{\alpha^2-x^2}},$$

and using the identity:

$$T_{n+1}(x) - xT_n(x) = -\sqrt{1-x^2}\,U_n(x),$$

we have

$$\cos(n\phi+\xi) = \frac{((\sqrt{\alpha^2-1})-\alpha)xT_n(x)+\alpha T_{n+1}(x)}{\sqrt{\alpha^2-x^2}}. \quad (9)$$

Hence, the function $\cos^2(n\phi+\xi)$ will be rational. Since this function is ≥ 0 for $-\alpha<x<\alpha$, the appropriate specification for the insertion gain, utilizing this function is:

$$|s_{12}|^2 = \frac{1}{1+\epsilon^2\cos^2(n\phi+\xi)}. \quad (10)$$

This gives a band-pass type of response of the type shown in Fig. 3, which is applicable in this case when $q=1$.

III. Multiple Stub Filter Functions

Although the case explicitly treated here was that of the dc zero of transmission of the insertion gain (i.e., a shunt short-circuited stub or a series open-circuited stub), the technique is general in nature and may be extended to other cases of interest by an appropriate transformation. Thus we may extend the idea to obtain a low-pass Chebyshev transmission line filter with a multiple zero of transmission in $|s_{12}(j\Omega)|$ as $\Omega\to\infty$. In such a case the gain function for a filter with equal load and generator terminations must be chosen to be of the form

$$|s_{12}(j\Omega)|^2 = \frac{(1+\Omega^2)^n}{P_{n+q}(\Omega^2)}. \quad (11)$$

P_{n+q} is a polynomial of order $2(n+q)$ in Ω with constant term unity so that there is no mismatch loss at dc, and (11) exhibits the zero at $\Omega=\infty\,(\theta=\pi/2)$ of order q in $|s_{12}(j\Omega)|$. Since we are discussing the low-pass case we let $y=\alpha\sin\theta$ ($y=0$, and $\theta=\beta l=0$ occur at dc in the middle of the pass band). Then (11) becomes

$$|s_{12}(j\Omega)|^2 = \frac{1}{\cos^{2n}\theta P_{n+q}\left(\frac{\sin^2\theta}{\cos^2\theta}\right)} = \frac{1}{1+\frac{H_{n+q}(y^2)}{(\alpha^2-y^2)^q}}, \quad y=\alpha\sin\theta. \quad (12)$$

As $y\to\alpha$, $|s_{12}|$ has a zero of order q.

We now wish to choose the polynomial $H_{n+q}(y^2)$ so as to obtain Chebyshev behavior in the pass band. Consider the function

$$f(y) = \cos(n\phi+q\xi) \quad (13)$$

where

$$\cos\phi = \alpha\sin\theta = y \quad (14)$$

and as in (8a) and (8b)

$$\cos\xi = \sqrt{\frac{1+\cos\delta}{2}} = y\sqrt{\frac{\alpha^2-1}{\alpha^2-y^2}} \quad (15a)$$

$$\sin\xi = \sqrt{1-\cos^2\xi} = \alpha\sqrt{\frac{1-y^2}{\alpha^2-y^2}}. \quad (15b)$$

Then

$$\cos(n\phi+q\xi) = \cos n\phi\cos q\xi - \sin n\phi\sin q\xi \quad (16)$$

and with

$$\cos n\phi\cos q\xi = \frac{T_n(y)R_q(y)}{\sqrt{(\alpha^2-y^2)^q}} \quad (17)$$

$$\sin n\phi\sin q\xi = \frac{\alpha(1-y^2)Q_{n-1}(y)M_{q-1}(y)}{\sqrt{(\alpha^2-y^2)^q}} \quad (18)$$

where $R_k(y)$, and $M_k(y)$ are polynomials of degree k in y, we may readily deduce that

$$|s_{12}(y)|^2 = \frac{1}{1+\epsilon^2\cos^2(n\phi+q\xi)} \quad (19)$$

is an even rational low-pass gain function in y with the appropriate zero of transmission of order $2q$ at $y=\alpha$, $\theta=\pi/2$, and $\Omega=\infty$.

Similarly, an alternate representation of a low-pass function with a multiple zero at $\Omega=\infty\,(\beta l=\pi/2)$ has the form

$$|s_{12}(y)|^2 = \frac{1-\epsilon^2}{1+\epsilon^2\cos(2n\phi+q\delta)} \quad (20)$$

where it is most convenient to determine the cosine term as

$$\cos(2n\phi + q\delta) = 2\cos^2(n\phi + q\xi) - 1 \tag{21}$$

since $\xi = \delta/2$. Hence, $|s_{12}(y)|^2$ of (20) may also be written

$$|s_{12}(y)|^2 = \frac{1}{1 + \dfrac{2\epsilon^2}{1-\epsilon^2}\cos^2(n\phi + q\xi)}\,. \tag{22}$$

A typical realization of either (20) or (22) is shown in Fig. 1. Here the zeros of transmission are achieved by series short-circuited stubs which present open circuits at $\theta = \pi/2$, and open-circuited shunt stubs which present short circuits at $\theta = \pi/2$. In Fig. 1 Kuroda's identity is used so that all stubs are open-circuited *shunt* stubs.

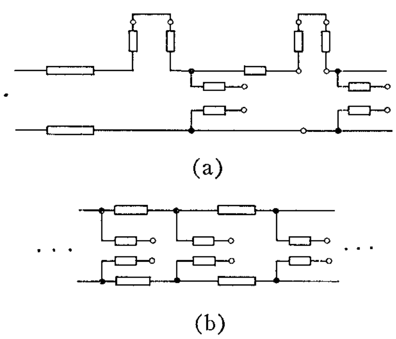


Fig. 1. Low-pass filter with multiple zero at $\beta l = \pi/2$. (a) Realization with series and shunt stubs. (b) Shunt stubs only by Kuroda identity.

The type of frequency response characteristic obtained from (22) is indicated in Fig. 2.

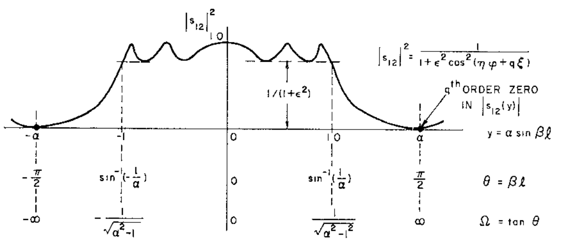


Fig. 2. Chebyshev low-pass response with higher order zero at $\theta = \pi/2 (n=3,\ q=2)$.

In a manner similar to that discussed for the low-pass case we may deduce a Chebyshev response functions for a multiple stub band-pass filter or transformer with a higher order zero of transmission at dc. Thus the available gain in the Ω plane is easily inferred by extending (1) to give

$$|s_{12}(j\Omega)|^2 = \frac{\Omega^{2q}(1+\Omega^2)^n}{P_{n+q}(\Omega^2)}, \qquad 0 \le |s_{12}|^2 \le 1. \tag{23}$$

This is realized by n cascaded lines and q stubs, but stubs must be both series and shunt connected with respect to the lines. The gain function of (23) has a zero of transmission of order $2q$ at $\Omega = 0$.

Let

$$x = \alpha \cos\theta \tag{24}$$

then as a function of x, and paralleling (3) we obtain

$$|s_{12}|^2 = \frac{1}{1 + \dfrac{H_{n+q}(x^2)}{(\alpha^2 - x^2)^q}}\,. \tag{25}$$

An appropriate choice of the polynomial H_{n+q} is obtained from the expression

$$\frac{H_{n+q}(x^2)}{(\alpha^2 - x^2)^q} = \epsilon^2\cos^2(n\phi + q\xi) \tag{26}$$

where, in the usual fashion,

$$\cos\phi = x \tag{27a}$$

$$\cos\xi = x\sqrt{\frac{\alpha^2-1}{\alpha^2-x^2}}\,. \tag{27b}$$

Thus

$$|s_{12}|^2 = \frac{1}{1+\epsilon^2\cos^2(n\phi + q\xi)}\,. \tag{28}$$

Of course, a form similar to (20) may also be employed.

It is readily shown that $|s_{12}|^2$, as given by (28) is rational, realizable, and has the appropriate qth order zero of transmission at $x = \pm\alpha$, or at both $\theta = 0,\ \pi$. A plot of a typical characteristic is shown in Fig. 3. A variety of scales are shown corresponding to the different variables used, but the variable proportional to real frequency is $\theta = \beta l = \omega l/c$, where c is the phase velocity of the wave on the transmission line. The response in θ exhibits the higher order zero of transmission at *both* cutoff points of the pass band, since the response is symmetric in θ about the quarter-wavelength point.

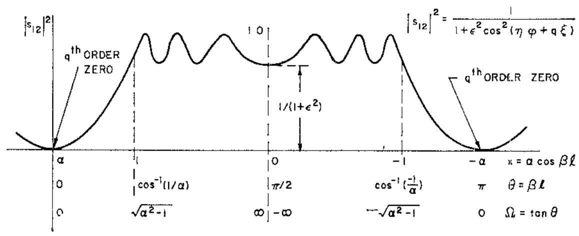


Fig. 3. Chebyshev band-pass response with higher order zero at $\theta = 0,\ \pi$; $(n=3,\ q=3)$.

A filter structure corresponding to $n=3$, $q=3$ (3 cascaded lines, 3 stubs) is shown in Fig. 4. In this case the Kuroda identity does not apply, and while it is possible to relocate the stubs at different points along the lines, the series stubs cannot be eliminated. In other words, a multiplicity of only *shunting short-circuited* stubs still produces only a *simple* zero of transmission in s_{12} at the origin.

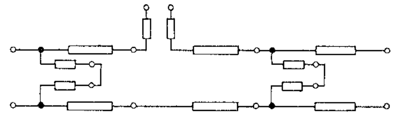


Fig. 4. Band-pass multistub filter.

In order to apply the band-pass filter function we must determine α, n, q, and ϵ from the specifications of the problem. The quantity α is determined by the bandwidth requirement as in the calculation of a cascaded line filter. Thus, we find the common line length is

$$l = \frac{1}{2}\frac{\lambda_H \lambda_L}{\lambda_H + \lambda_L} \tag{29a}$$

where λ_H and λ_L are wavelengths at the high and low frequency band edges. Then in

$$\beta l = \theta = \frac{2\pi l}{\lambda} \tag{29b}$$

so that using (24) with $x=1$ at $\lambda=\lambda$

$$\alpha = \frac{1}{\cos \dfrac{2\pi l}{\lambda_L}} \cdot \tag{30}$$

To determine the pass band tolerance, refer to Fig. 3, and note that

$$10 \log (1 + \epsilon^2) = L_P \tag{31}$$

where L_P is maximum insertion loss in the pass band in decibles.

In order to compute the number of lines and stubs (n and q) we use the required stop-band insertion loss and approximate (28) by assuming $x>1$. In this case we have

$$T_n(x) \approx 2^{n-1}x^n, \qquad x > 1 \tag{32a}$$

$$U_n(x) \approx \sqrt{1 - x^2}\, 2^{n-1}x^{n-1}, \qquad x > 1. \tag{32b}$$

Then we may approximate

$$\cos^2 (n\phi + q\xi) \approx \frac{1}{4}(2x)^{2(n+q)}\left[\frac{\alpha^2 - 1}{\alpha^2 - x^2}\right]^q \quad x > 1. \tag{33}$$

In any physical transmission line filter it is usual to separate the stubs by lines, otherwise a number of stubs must be interconnected with zero line separations, an impractical procedure. Hence, $q \leq n+1$. If we take $q=n$ in (33)

$$\cos^2 (n\phi + q\xi) \approx \frac{1}{4}\left[(2x)^4 \frac{\alpha^2 - 1}{\alpha^2 - x^2}\right]^n, \; x > 1, q = n. \tag{34}$$

It is now a simple matter to approximately determine an integer n for a prescribed stop-band loss. This can be checked subsequently from the exact expression and if necessary modified by adding or subtracting one line or stub.

As an example of the procedure suppose we consider a band-pass filter with

1) $f_L=2000$ Mc/s, $f_H=3000$ Mc/s.
2) Maximum pass band loss 0.16 dB.
3) At 1000 Mc/s response to be down approximately 50 dB.

Then

$$10 \log (1 + \epsilon^2) = 0.16$$

$$\epsilon = 0.2.$$

The common line length is (29a)

$$l = \frac{1}{2}\frac{10 \times 15}{10 + 15} = 3 \text{ cm}.$$

Thus

$$\alpha = \frac{1}{\cos 2\pi \dfrac{3}{15}} = \frac{1}{\cos 72^\circ} = 3.24.$$

At 1000 Mc the value of x is

$$x = \alpha \cos \theta = 3.24 \cos 2\pi \frac{3}{30}$$

$$= 2.62$$

and, because of the symmetry of the characteristic, this loss also occurs at $x=-2.62$ or $\theta=144°$, $f=4000$ Mc.

Using (28), and neglecting the unity term in the denominator, the stop-band loss is

$$L_S = 10 \log \epsilon^2 \cos^2 (n\phi + q\xi)$$

and for our problem, (34) gives

$$\epsilon^2 \cos^2 (n\phi + q\xi) = \frac{0.04}{4}\left[(2 \times 2.62)^4 \frac{3.24^2 - 1}{3.24^2 - 2.62^2}\right]^n$$

$$= 0.01[3000]^n$$

and with

$$n = q = 2$$

we get

$$10 \log \frac{1}{|s_{12}|^2} = 10 \log 9 \times 10^4 = 49.6 \text{ dB},$$

so that two lines and two stubs are a reasonable choice for the prescribed specifications. We may compare results for the stop-band loss when only one stub is used by using (33) with $q=1$. Thus, for $\epsilon=0.2$

At 1000 Mc/s and 4000 Mc/s, $x=2.62$:

$n=2$, $q=2$;	$L_S=49.6$ dB
$n=3$, $q=1$;	$L_S=42.0$ dB
$n=4$, $q=1$;	$L_S=52.0$ dB

IV. Adjustment of Load by Partial Stub Extraction

Band-pass filters and transformers which utilize a single short-circuited shunt stub in addition to cascaded lines are important from a practical point of view. This type of design is useful in that no open-circuited stubs are required (an open circuit may be difficult to realize at high frequencies) and series interconnected stubs, which are generally impractical at high frequencies, are also eliminated. The single shorting stub produces a simple zero of transmission at the origin and provides the means for obtaining a band-pass filter with reasonable cutoff characteristics and dc response suppressed.

The main difficulty in designing the single stub band-pass filter or transformer is that, if conventional methods of synthesis are employed, the load cannot be prescribed in advance. This is due to the zero of transmission at dc. In contrast to this, a low-pass configuration which only employs cascaded lines is transparent at dc and the input impedance is precisely the load impedance at $\omega=0$. When the shorting stub is present, the input impedance at $\omega=0$ is zero regardless of the load; hence, the dc impedance condition yields no direct information. In order to handle this problem let us first examine the effect of synthesizing the network by changing the point along the cascade chain at which the stub is placed. We will show that the terminating load resistance decreases monotonically as the placement of the stub is postponed to points along the chain further removed from the input.

Consider Fig. 5(a). Here the stub is located at the front of a line section. Since a stub (all lines and stubs of same length l and propagation constant β) acts like an inductance in the λ domain. (λ, not to be confused with wavelength, is the complex variable which continues $\Omega=\tan\beta l$ into the complex frequency domain [1], [2]. At real frequencies in the transformed variable Ω, $\lambda=j\Omega$. More generally $\lambda=\Sigma+j\Omega$, corresponding to $p=\sigma+j\omega$.) The input impedance z is given as the parallel combination of L_1 and z_1

$$z(\lambda) = \frac{L_1\lambda z_1(\lambda)}{L_1\lambda + z_1(\lambda)} = \frac{L_1\lambda}{1+\dfrac{L_1}{z_1(\lambda)}\lambda}\,. \tag{35}$$

Expanding about $\lambda=0$

$$z(\lambda) = L_1\lambda - \frac{L_1^2\lambda^2}{z_1(0)} + \cdots = a_1\lambda + a_2\lambda^2 + \cdots. \tag{36}$$

But with the stub eliminated it is clear that $z_1(0)=R_{L1}$ so that

$$z_1(0) = -\frac{a_1^2}{a_2} = R_{L1}. \tag{37}$$

Suppose now we first extract a line before we remove the stub. Then, as in Fig. 5(b), a line is extracted starting at terminal plane $x-x$ and the stub is located at $y-y$. Richards' formula [2] may be used to obtain the impedance $z^{(1)}$ at the end of the extracted line in terms of the properties of z. Thus

$$z^{(1)}(\lambda) = z(1)\frac{z(\lambda)-\lambda z(1)}{z(1)-\lambda z(\lambda)}\,. \tag{38a}$$

Since $z(\lambda)$ is realizable, hence positive real, $z(1)=\sigma>0$. Thus in the neighborhood of $\lambda=0$ using (36) in (38a)

$$z^{(1)}(\lambda)\big|_{\lambda\to 0} = \sigma\frac{(a_1-\sigma)\lambda + a_2\lambda^2+\cdots}{\sigma - a_1\lambda^2 - a_2\lambda^3}\,. \tag{38b}$$

Higher order terms may be disregarded in the denominator with respect to σ, hence

$$z^{(1)}(\lambda)\big|_{\lambda\to 0} = (a_1-\sigma)\lambda + a_2\lambda^2 + \cdots = b_1\lambda + b_2\lambda^2 + \cdots.$$

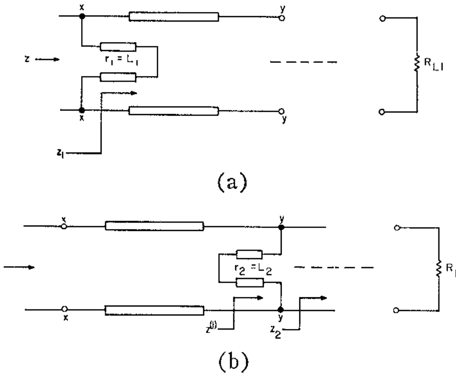


Fig. 5. Effect of stub location $R_{L1}\geq R_L\geq R_{L2}$. (a) Stub extracted in front of a line. (b) Stub extracted at end of a line.

The impedance $z^{(1)}(\lambda)$ is positive real, hence at the simple zero $\lambda=0$ the leading coefficient must be positive

$$b_1 = (a_1-\sigma) > 0. \tag{39}$$

As before we may evaluate the load impedance as $R_{L2}=z_2(0)$ and applying the method of (37) and substituting from (36)

$$z_2(0) = -\frac{b_1^2}{b_2} = \frac{(a_1-\sigma)^2}{a_2} = \frac{(L_1-\sigma)^2}{L_1^2}R_{L1} = R_{L2} \tag{40a}$$

and since $(a_1-\sigma)=(L_1-\sigma)>0$, then $(L_1-\sigma)^2<L_1^2$ and

$$R_{L2} < R_{L1}. \tag{40b}$$

Thus the final load resistance decreases monotonically as the terminal plane for stub extraction is chosen further from the generator. Since [see (36)]

$$b_1 = L_2 = (a_1-\sigma) = (L_1-\sigma)$$

(40a) also yields

$$\frac{L_2^2}{L_1^2} = \frac{R_{L2}}{R_{L1}}, \qquad L_2 < L_1. \tag{41}$$

We may summarize the results thus far by expressing the pertinent quantities in terms of the input impedance $z(p)$. Thus:

$$L_1 = a_1 = z'(0) \tag{42a}$$

$$L_2 = L_1 - \sigma = z'(0) - z(1) \tag{42b}$$

$$R_{L1} = -\frac{a_1^2}{a_2} = -\frac{2[z'(0)]^2}{z''(0)} \tag{42c}$$

$$R_{L2} = \left[\frac{L_2}{L_1}\right]^2 R_{L1} = -2\frac{[z'(0)-z(1)]^2}{z''(0)}\,. \tag{42d}$$

Thus the stub characteristic impedances L_1, L_2 and the final load values R_{L1}, R_{L2} may be computed in advance from the input impedance.

Suppose that the desired value of terminating load resistance $\hat{R}_{L2}$ lies between the values R_{L1}, R_{L2} of (42c) and (42d), $R_{L2} < \hat{R}_{L2} < R_{L1}$. We may extract only a portion of the stub at the front, say $\bar{L}$, then extract an additional stub $\hat{L}_2$ at the end of the line and achieve the desired resistance value $\hat{R}_{L2}$. Note that the two shunt stubs do not change the order of the simple zero at the origin.

To see how we may predict $\bar{L}$ in advance for a prescribed load, we assume a given impedance $z(\lambda)$ with L_1, L_2, R_{L1}, and R_{L2} known according to (42). Now suppose $\bar{L}$ is removed at the front. Then the remaining stub, $\hat{L}_1$ which combines in parallel with $\bar{L}$ at the front to give L_1 has characteristic impedance

$$L_1 = \frac{L_1 \bar{L}}{\bar{L}_1 - L_1}. \tag{43}$$

The impedance looking in at $\hat{L}_1$ is

$$\hat{z}(\lambda) = \frac{\bar{L}\lambda z(\lambda)}{L\lambda - z(\lambda)}. \tag{44}$$

If, instead of removing $\hat{L}_1$ at the front we defer the removal of this stub reactance ($\bar{L}$, the partial reactance is presumed extracted) to the rear of the line section then a stub $\hat{L}_2$ will be available at the end of the line section. By (42b)

$$L_2 = L_1 - \hat{z}(1) \tag{45a}$$

and using (43), (44), and (42b)

$$\hat{L}_2 = \bar{L}^2 \frac{L_1 - z(1)}{(\bar{L} - L_1)(\bar{L} - z(1))} = \frac{\bar{L}^2 L_2}{(\bar{L} - L_1)(\bar{L} - z(1))}. \tag{45b}$$

Then the new load resistance

$$R_{L2} < \hat{R}_{L2} = \left(\frac{\hat{L}_2}{\hat{L}_1}\right)^2 R_{L1}$$

$$\hat{R}_{L2} = \left(\frac{\bar{L}}{\bar{L} - z(1)}\right)^2 \cdot \left(\frac{L_2}{L_1}\right)^2 R_{L1}$$

and using (42d)

$$\hat{R}_{L2} = \left(\frac{\bar{L}}{\bar{L} - z(1)}\right)^2 R_{L2}. \tag{46}$$

Solving for $\bar{L}$:

$$\bar{L} = \frac{z(1)}{1 - \sqrt{\dfrac{R_{L2}}{\hat{R}_{L2}}}} > 0. \tag{47}$$

Thus with $\hat{R}_{L2}$ prescribed, satisfying

$$R_{L2} < \hat{R}_{L2} < R_{L1}$$

one can easily calculate the characteristic impedance $\bar{L}$ when a partial stub extraction is performed.

As an example which is not intended to be practical, but which merely illustrates the procedure, consider the admittance

$$y(\lambda) = \frac{10}{\lambda} + \frac{0.01 + \lambda}{1 + 0.01\lambda}.$$

It is clear that the total stub extraction yields

$$\frac{1}{L_1} = 10$$

$$\frac{1}{L_2} = \frac{\dfrac{1}{L_1}}{1 - \dfrac{1}{y(1)L_1}} = \frac{10}{1 - \dfrac{10}{11}} = 110$$

$$\frac{1}{R_{L1}} = \frac{d}{d\lambda}\left[10 + \lambda \frac{0.01 + \lambda}{1 + 0.01\lambda}\right]_{\lambda \to 0}$$

$$= \frac{d}{d\lambda}\left[\lambda \frac{0.01 + \lambda}{1 + 0.01\lambda}\right]_{\lambda \to 0}.$$

This derivative is merely the leading term of the power series obtained by expanding at $\lambda = 0$

$$\lambda \frac{0.01 + \lambda}{1 + 0.01\lambda} = 0.01\lambda + (1 - 10^{-4})\lambda^2 + \cdots.$$

Therefore

$$\frac{1}{R_{L2}} = 0.01, \qquad R_{L1} = 100. \tag{48a}$$

Also

$$R_{L2} = \left(\frac{L_2}{L_1}\right)^2 R_{L1} = \left(\frac{10}{110}\right)^2 100 = 0.825\Omega. \tag{48b}$$

Hence

$$0.825 < \hat{R}_{L2} < 100$$

which represents a very broad range of possible values. Suppose we wish $\hat{R}_{L2} = 1$. Then by (47)

$$\bar{L} = \frac{z(1)}{1 - \sqrt{\dfrac{R_{L2}}{\hat{R}_{L2}}}} = \frac{1}{11\left(1 - \dfrac{10}{11}\right)} = 1\Omega \tag{49a}$$

thus, the stub in front has unit characteristic impedance. Then the stub at the end of the line is (45b)

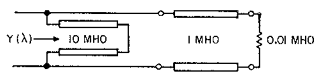
$$L_2 = \frac{\bar{L}^2 L_2}{(\bar{L} - L_1)(\bar{L} - z(1))} = \frac{1}{110\left(1 - \dfrac{1}{10}\right)\left(1 - \dfrac{1}{11}\right)}$$

$$= \frac{1}{90}\,\Omega. \tag{49b}$$

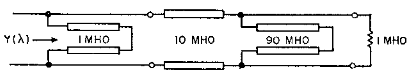
The line separating the two stubs has characteristic admittance

$$\mathcal{Y}(1) = \frac{1}{\mathcal{Z}(1)} = \left[\frac{10}{\lambda} - \frac{1}{\lambda} + \frac{0.01 + \lambda}{1 + 0.01\lambda}\right]_{\lambda=1}$$
$$= 10 \text{ mho.}$$

Figure 6(a) shows the realization of the given impedance function with the stub immediately extracted, and Fig. 6(b) shows the synthesis for a one-ohm load.



(a)



(b)

Fig. 6. Control of load resistance by stub location. (a) Realization of $Y(\lambda)$ with maximum load impedance. (b) Realization of $Y(\lambda)$ for one-ohm termination.

V. Realizability of a Prescribed Load Resistance

Section IV has shown that the load resistance can be adjusted over a wide range by choosing the point at which the extraction of the shunt short-circuiting stubs takes place. In applying this technique to the synthesis of a single stub filter function, it would be important to determine the bounds on the load resistance before the actual synthesis is carried out, to avoid needless factorization calculations. The highest value of load resistance occurs when the stub is at the front of a cascaded line filter, and the lowest bound when the stub is across the load. If the prescribed load resistance falls within these bounds we may proceed with the filter synthesis and, as each cascaded line is removed, compute [(42c) and (42d)] the bounds on R_L with the stub at the front, or at the rear of the following line section. At the particular line section where the prescribed load falls between the two computed values, the partial stub extraction method described previously is used to realize exactly the given terminating resistance.

It is the purpose of this section to determine bounds on realizable terminating load resistance from the *prescribed insertion gain function* of the cascaded line network when this function has a simple zero of transmission at the origin in the λ plane. One wishes to know the bounds without proceeding with the detailed synthesis.

Referring to Fig. 7, the input admittance with the stub at the front of the filter has the form

$$y(\lambda) = \frac{Y_S}{\lambda} + y_1 \tag{50}$$

where Y_S is the characteristic admittance of the stub, and the other quantities are as indicated on Fig. 7.

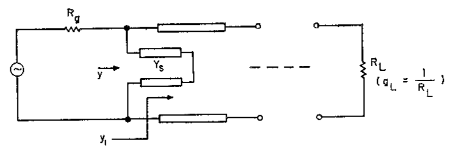


Fig. 7. Schematic of single stub-cascaded line system.

Then

$$\text{Re } y(0) = g_L. \tag{51}$$

At any frequency, the input conductance R_g Re $y(j\Omega)$, normalized to the generator impedance R_g, is related to the scattering coefficients of the stub-cascaded line network (normalized to R_g and R_L) by [16]

$$R_g \text{ Re } y = \frac{1 - |s_{11}|^2}{|1 + s_{11}|^2} = \frac{|s_{12}|^2}{|1 + s_{11}|^2}. \tag{52}$$

The second relation of (52) follows because the transmission line network is lossless.

Now, by (51)

$$R_g \text{ Re } y\big|_{\lambda\to 0} = R_g g_L. \tag{53}$$

Hence, with

$$s_{11} = \frac{1 - R_g y}{1 + R_g y}$$

we have

$$|1 + s_{11}|^2 = \left|1 + \frac{1 - R_g y}{1 + R_g y}\right|^2 = \frac{4}{|1 + R_g y|^2}$$

and using (50) with $y_1(0) = g_L$, and $\lambda = j\Omega$

$$|1 + s_{11}|^2_{\lambda\to 0} = \frac{4}{(1 + R_g g_L)^2 + \dfrac{R_g^2 Y_S^2}{\Omega^2}}\Bigg|_{\Omega\to 0}$$
$$= \frac{4\Omega^2}{R_g^2 Y_S^2}\Bigg|_{\Omega\to 0} \tag{54}$$

In order to utilize (52) to determine a constraint on the load resistance, we must presume a form for $|s_{12}|^2$, the available gain function of the system. Let us use the single stub band-pass function of (6). Then applying (3)

$$|s_{12}|^2 = \frac{1 - \epsilon^2}{1 + \epsilon^2 \cos(2n\phi + \delta)} = \frac{1 - \epsilon^2}{1 + \dfrac{\epsilon^2 f_1(x^2)}{\alpha^2 - x^2}} \tag{55}$$

where $\epsilon^2 f_1(x^2) = H_{n+1}(x^2)$ is an even polynomial in x. Hence, using

$$\Omega^2 = \frac{\sin^2\theta}{\cos^2\theta} = \frac{\alpha^2(1 - \cos^2\theta)}{x^2} = \frac{\alpha^2 - x^2}{x^2} \tag{56}$$

where θ and x are defined by (29b) and (24), respectively, we obtain

$$\left.\frac{|s_{12}|^2}{|1+s_{11}|^2}\right|_{\substack{\Omega\to 0\\ x\to\alpha}} = R_g g_L = \left.\frac{1-\epsilon^2}{1+\dfrac{\epsilon^2 f_1(\alpha^2)}{\alpha^2-x^2}}\cdot\frac{R_g^2 Y_S^2}{4\dfrac{\alpha^2-x^2}{\alpha^2}}\right|_{x\to\alpha}$$

or

$$g_L = \frac{(1-\epsilon^2)\alpha^2 R_g Y_S^2}{4\epsilon^2 f_1(\alpha^2)} \tag{57a}$$

and

$$f_1(\alpha^2) = (\alpha^2 - x^2)\cos(2n\phi + \delta)\,|_{x=\alpha}. \tag{57b}$$

If the gain function is of the form of (10), then (57a) becomes

$$g_L = \frac{\alpha^2 R_g Y_S^2}{4\epsilon^2 f_2(\alpha^2)} \tag{57c}$$

with

$$f_2(\alpha^2) = (\alpha^2 - x^2)\cos^2(n\phi + \xi)\,|_{x=\alpha}. \tag{57d}$$

Equations (57a) and (57c) are essentially identical ($\epsilon^2 \ll 1$) except for f_1 and f_2.

We thus have a relation between g_L and the stub characteristic admittance Y_S. In order to solve for g_L in terms of known parameters [i.e., ϵ, α, $f(\alpha^2)$ and R_g] we must seek an additional relation. This is obtained from gain-bandwidth theory.

Consider a generator R_g feeding a lossless two-port terminated in $R_L = 1/g_L$. At the input, an inductance L is in shunt with the two-port. This emulates Fig. 7 in the λ plane with $L = Y_S$.

Suppose the scattering parameters of the lossless two-port (including the shunt stub) are $s_{11}(\lambda)$, $s_{22}(\lambda)$, $s_{12}(\lambda) = s_{21}(\lambda)$, normalized to R_g at port one and R_L at port two. Then, because of the stub of shunt reactance $= L\Omega = Y_S^{-1}\Omega$, the following integral relation must be satisfied [17], [18]

$$\int_0^\infty \frac{1}{\Omega^2}\ln\frac{1}{|s_{11}|^2}\,d\Omega = \frac{2\pi}{R_g Y_S} - P. \tag{58}$$

If the lossless two-port only contains commensurate transmission lines then $s_{11}(\lambda)$ is a rational function and in the preceding expression P is given by

$$P = \sum_k \frac{1}{\lambda_K} > 0 \tag{59}$$

where λ_K are the right-half plane zeros of $s_{11}(\lambda)$, and P is a positive real quantity since the zeros have positive real parts and are real or occur in conjugate pairs.

The integral expression (58) may be expressed in terms of the prescribed insertion gain function $|s_{12}(j\Omega)|^2$ since for a lossless two-port

$$1 - |s_{12}(j\Omega)|^2 = |s_{11}(j\Omega)|^2$$

Thus the constraint of (58) becomes

$$Q = \int_0^\infty \frac{1}{\Omega^2}\ln\frac{1}{1-|s_{12}(j\Omega)|^2}\,d\Omega = \frac{2\pi}{R_g Y_S} - P \tag{60}$$

and

$$Y_S = \frac{2\pi}{R_g(Q+P)}. \tag{61}$$

We may substitute this in (57a) or (57b) to determine $R_L = 1/g_L$

$$R_L = R_g\frac{\epsilon^2 f_1(\alpha^2)(Q+P)^2}{(1-\epsilon^2)\alpha^2\pi^2} = CR_g(Q+P)^2 \tag{62a}$$

and

$$R_L = R_g\frac{\epsilon^2 f_2(\alpha^2)(Q+P)^2}{\alpha^2\pi^2} \tag{62b}$$

where C as defined by (62a) or (62b) is therefore only a function of the gain characteristic.

Thus with the stub immediately *extracted* at the generator end of the network, the ratio of load to generator resistance is

$$\frac{R_{L1}}{R_g} = C(Q+P)^2 \tag{63}$$

where R_{L1} is the load obtained by synthesizing the network by first removing the stub.

In order to determine the effect of extracting the stub last, i.e., at the load end of the network, we may use the same gain function $|s_{12}|^2$, but we must synthesize the cascaded line network from the back end and adjust the back end resistance so that the generator resistance which now plays the role of termination is R_g. The only effect of this is to interchange R_L and R_g in (63) so that

$$\frac{R_g}{R_{L2}} = C(Q+P)^2 \tag{64}$$

with R_{L2} the terminating resistance when the stub is removed *at the load end* of the cascaded chain, and Q and P as previously defined.

The final problem in using these relations to obtain the bounds on R_L is to ascertain the value of P. This parameter is not unique, but depends on the choice of zeros for $s_{11}(\lambda)$.

The prescribed available gain function is $s_{12}(\lambda)s_{12}(-\lambda)$. The function $s_{11}(\lambda)$ is obtained by factoring $s_{11}(\lambda)s_{11}(-\lambda) = 1 - s_{12}(\lambda)s_{12}(-\lambda)$. In this factorization process the poles (denominator zeros) of $s_{11}(\lambda)$ must be in the left-half λ plane, but the zeros may be distributed in a variety of ways. These zeros will occur as image pairs about $j\Omega$ in the left- and right-half planes, and will be either real or occur in complex conjugate pairs. Thus the zeros of $s_{11}(\lambda)s_{11}(-\lambda)$ will be symmetrically located with respect to both the Σ and $j\Omega$ axes. If the degree of $s_{11}(\lambda)s_{11}(-\lambda)$ is $2n$ we may select any n of the zeros for $s_{11}(\lambda)$ provided: a) they form conjugate pairs; b) if a zero λ_k is selected its image zero with respect to the $j\Omega$ axis is not used.

Let us suppose that any permissible set of n roots is chosen for the numerator of $s_{11}(\lambda)$ from the $2n$ roots of $s_{11}(\lambda)s_{11}(-\lambda)$. Denote this choice as p_1, and the value of P (59) as P_1. A complementary set of roots p_2 is determined with all the left- and right-half plane roots of p_1 interchanged. This gives P the value P_2. Suppose that $P_1 \geq P_2$. Further, let the value of P be P_0 when all the n roots are chosen in the right-half plane. Clearly this is the largest value of P and

$$P_1 + P_2 = P_0. \tag{65}$$

Now when the stub is at the head end of the chain, (63) applies. Let the two values of (R_{L1}/R_g) for the complementary choice of roots p_1 and p_2 be r_{L1}, r_{L2}.

$$\frac{R_{L1}}{R_g} = r_{L1} = C(Q + P_1)^2 \quad (\text{zeros } p_1 \text{ in } s_{11}(\lambda)) \tag{66a}$$

$$\frac{R_{L1}}{R_g} = r_{L2} = C(Q + P_2)^2 \quad (\text{zeros } p_2 \text{ in } s_{11}(\lambda)). \tag{66b}$$

The parameter Q only depends on the prescribed gain function so it is the same in both (66a) and (66b).

Similarly, let r_{L3}, r_{L4} correspond to the normalized load resistance values (R_{L2}/R_g) when the stub is removed at the load termination of the transmission line chain.

If the input scattering coefficient of the chain is $s_{11}(\lambda)$ normalized to R_g and the back end scattering coefficient is $s_{22}(\lambda)$ normalized to R_L, then the unitary requirement on $j\Omega = \lambda$ imposes

$$s_{22}(\lambda) = -\, s_{11}(-\lambda)\, \frac{s_{12}(\lambda)}{s_{12}(-\lambda)} \cdot \tag{67}$$

The function $s_{12}(\lambda)$, as determined from the gain function form of (1) must be

$$s_{12}(\lambda) = \pm \frac{\lambda(1 - \lambda^2)^{n/2}}{D_n(\lambda)} \tag{68}$$

where $D_n(\lambda)$ is a Hurwitz polynomial of degree n in λ. Then denoting

$$s_{11} = \pm \frac{N_{11}(\lambda)}{D_n(\lambda)} \tag{69}$$

we deduce from (67)

$$s_{22}(\lambda) = \pm \frac{N_{11}(-\lambda)}{D_n(\lambda)} \cdot \tag{70}$$

There is actually no ambiguity of sign since the presence of a stub forces the input reflection factor at dc to be that of a short circuit, hence the sign in (70) is chosen so that

$$s_{22}(0) = -1. \tag{71}$$

The consequence of (70) is that if a set of zeros is chosen for $s_{11}(\lambda)$, then the complementary set appears in $s_{22}(\lambda)$. Hence if the set p_1 is chosen for $s_{11}(\lambda)$, we get p_2 in $s_{22}(\lambda)$ and when the stub is at the rear (64) gives

$$r_{L3} = \frac{1}{C(Q + P_2)} = \frac{1}{r_{L2}} \quad (\text{zeros } p_1 \text{ in } s_{11}). \tag{72a}$$

Similarly

$$r_{L4} = \frac{1}{C(Q + P_1)} = \frac{1}{r_{L1}} \quad (\text{zeros } p_2 \text{ in } s_{11}). \tag{72b}$$

These results are summarized in the following table.

Stub Location	Roots of $s_{11}(\lambda)$	$r_L = R_L/R_g$	Roots of $s_{22}(\lambda)$
Front	p_1	$r_{L1} = C(Q+P_1)^2$	p_2
Front	p_2	$r_{L2} = C(Q+P_2)^2$	p_1
Back	p_1	$r_{L3} = \dfrac{1}{r_{L2}}$	p_2
Back	p_2	$r_{L4} = \dfrac{1}{r_{L1}}$	p_1

With $P_1 \geq P_2$ the table indicates that

$$r_{L1} \geq r_{L2} \tag{73a}$$

$$r_{L3} \geq r_{L4}. \tag{73b}$$

Furthermore, we have already shown that when the stub is located at the front, it gives rise to the largest value of terminating resistance for a prescribed reflection factor function. Hence with the p_1 zeros in $s_{11}(\lambda)$ we must also satisfy

$$r_{L1} \geq r_{L3} \tag{74a}$$

and with p_2 zeros in $s_{11}(\lambda)$

$$r_{L2} \geq r_{L4}. \tag{74b}$$

The absolute maximum for r_L must occur for the largest value of P, i.e., when all the zeros of $s_{11}(\lambda)$ are in the right-half plane, i.e., $P = P_0 = P_1$, $r_L = r_{L1}$. Then $P_2 = 0$ and, referring to (74b), the minimum value of $r_L = r_{L4}$. Thus

$$r_{L-\max} = C(Q + P_0)^2 \tag{75a}$$

$$r_{L-\min} = \frac{1}{C(Q + P_0)^2} = \frac{1}{r_{L-\max}} \cdot \tag{75b}$$

However, it may not be possible to realize all values of r_L between these limits. Two cases may be distinguished by referring to the inequality constraints of (73) and (74).

Case 1: $r_{L2} \geq 1$, $P_1 \geq P_2$. For the distribution of zeros p_1, corresponding to r_{L1} as in the table $r_{L1} \geq r_{L2} \geq 1$ and by the stub extraction technique we may defer the stub removal until the very end of the chain and obtain $r_{L1} \geq r_L \geq r_{L3}$. On the other hand, we may place the complementary zero set p_2 in $s_{11}(\lambda)$ and, by deferring stub extraction until the end of the chain, obtain $r_{L2} \geq r_L \geq r_{L4}$. As indicated in Fig. 8 we can therefore cover the entire range between r_{L1} and r_{L4}. Hence

$$\text{Case 1:} \quad r_{L2} \geq 1; \quad r_{L4} \leq r_L \leq r_{L1}. \tag{76a}$$

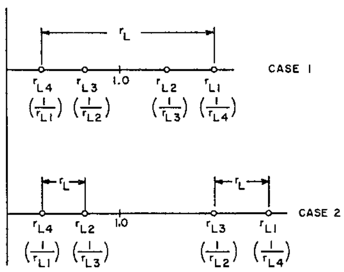


Fig. 8. Range of terminating load resistance.

Now suppose

Case 2: $r_{L2} \leq 1$. Then the inequalities demand the distribution shown for Case 2 in Fig. 8. Thus with all zeros p_1 in $s_{11}(\lambda)$, the stub in front gives r_{L1} and by stub extraction we can only cover the range $r_{L1} \geq r_L \geq r_{L3}$. With the complementary set of zeros p_2 in $s_{11}(\lambda)$ we cover the range $r_{L2} \geq r_L \geq r_{L4}$ and, since these ranges do not necessarily overlap, there may be gaps as shown in Fig. 8. Thus

$$\text{Case 2: } r_{L2} \leq 1; \; r_{L1} \geq r_L \geq r_{L3}, r_{L2} \geq r_L \geq r_{L4}. \tag{76b}$$

Of course, we always have the option of altering the zero distribution in $s_{11}(\lambda)$ to attempt to realize the continuous coverage of Case 1, Fig. 8.

It is also clear that a sufficient condition for obtaining all values of $r_{L-\min} \leq r_L \leq r_{L-\max}$ is that Case 1 apply when zeros corresponding to p_1 are all in the right-half plane, and p_2 is the complementary distribution without right-half plane zeros. Hence, a sufficient condition for maximum coverage is obtained from Case 1 as

$$r_{L-\min} \leq r_L \leq r_{L-\max} \quad \text{for} \quad CQ^2 \geq 1. \tag{77}$$

The quantity CQ^2 can be computed directly without factorization from the prescribed gain function, although numerical integration to obtain Q may be required. Furthermore, (77) is a sufficient condition that assures $r_L = 1$ is attainable, i.e., that a band-pass filter with a simple dc zero of transmission can be realized with load and generator resistance equal. More generally, if (77) applies, this is sufficient to assure the realization of a broadband transformer with any resistive termination between the values CQ^2 and $1/CQ^2$.

In order to aid in the application of (77), a rough approximation of C and Q may be derived. First note that if the gain function of (10) is used

$$|s_{12}|^2 = \frac{1}{1 + \epsilon^2 \cos^2(n\phi + \xi)} \tag{78a}$$

and if (55) is used

$$|s_{12}|^2 = \frac{1 - \epsilon^2}{1 + \epsilon^2 \cos(2n\phi + \delta)}$$

$$= \frac{1}{1 + \dfrac{2\epsilon^2}{1 - \epsilon^2} \cos^2(n\phi + \xi)}. \tag{78b}$$

Both of these equations have the form

$$|s_{12}|^2 = \frac{1}{1 + \gamma^2 \cos^2(n\phi + \xi)}. \tag{79}$$

In order to estimate Q of (60) it is necessary to approximate the integrand $1/\Omega^2 \ln 1/|s_{11}|^2$. In the pass band we may suppose that $|s_{12}|^2$ is flat and is represented by its average value. In this region $|s_{12}|^2$ oscillates between $1/1+\gamma^2$ and $1/1-\gamma^2$. Hence the average value of $|s_{12}|^2$ is

$$0 \leq x \leq 1 \quad |s_{12}|_{\text{Av.}}^2 = \frac{1}{2} \frac{2 + \epsilon^2}{1 + \epsilon^2} \quad \text{(Eq. 78a)}$$

$$0 \leq x \leq 1 \quad |s_{12}|_{\text{Av.}}^2 = \frac{1}{1 + \epsilon^2} \quad \text{(Eq. 78b).}$$

With $|s_{11}|^2 = 1 - |s_{12}|^2$ we obtain

$$0 \leq x \leq 1 \quad |s_{11}|_{\text{Av.}}^2 = \frac{\epsilon^2}{2(1 + \epsilon^2)} \cong \frac{\epsilon^2}{2} \quad \text{(Eq. 78b)} \tag{80a}$$

$$0 \leq x \leq 1 \quad |s_{11}|_{\text{Av.}}^2 = \frac{\epsilon^2}{1 + \epsilon^2} \cong \epsilon^2 \quad \text{(Eq. 78a).} \tag{80b}$$

Since the edge of the pass band occurs at $x = 1$, or $\Omega = \sqrt{\alpha^2 - 1}$ (see Fig. 3)

$$Q = \int_0^\infty \frac{1}{\Omega^2} \ln \frac{1}{|s_{11}|^2} d\Omega = \int_0^{\sqrt{\alpha^2-1}} \frac{1}{\Omega^2} \ln \frac{1}{|s_{11}|^2} d\Omega$$

$$+ \int_{\sqrt{\alpha^2-1}}^\infty \frac{1}{\Omega^2} \ln \frac{1}{|s_{11}|^2} d\Omega = Q_A + Q_B \tag{81}$$

and

$$Q_B \cong \int_{\sqrt{\alpha^2-1}}^\infty \frac{1}{\Omega^2} \ln \frac{1}{|s_{11}|_{\text{Av.}}^2} d\Omega$$

or

$$Q_B \cong \frac{1}{\sqrt{\alpha^2 - 1}} \ln \frac{2}{\epsilon^2} \quad \text{(Eq. (80a))} \tag{82a}$$

$$Q_B \cong \frac{1}{\sqrt{\alpha^2 - 1}} \ln \frac{1}{\epsilon^2} \quad \text{(Eq. (80b)).} \tag{82b}$$

To estimate Q_A we note that $\cos^2(n\phi + \xi) > 1$ in (79) for $1 < x \leq \alpha$. Thus

$$|s_{11}|^2 = 1 - |s_{12}|^2 = \frac{\gamma^2 \cos^2(n\phi + \xi)}{1 + \gamma^2 \cos^2(n\phi + \xi)}$$

$$\cong 1 - \frac{1}{\gamma^2 \cos^2(n\phi + \xi)} \quad 1 < x \leq \alpha.$$

Hence

$$\ln \frac{1}{|s_{11}|^2} \cong \frac{1}{\gamma^2 \cos^2(n\phi + \xi)} \quad 1 < x \leq \alpha. \tag{83}$$

Now use the approximation of (33) for $\cos^2(n\phi+\xi)$ with $q=1$

$$\cos^2(n\phi+\xi) = \frac{1}{4}(2x)^{2(n+1)}\left(\frac{\alpha^2-1}{\alpha^2-x^2}\right) = \frac{K^2\alpha^2}{\Omega^2(1+\Omega^2)^n} \tag{84}$$

where we have employed

$$\Omega^2 = \frac{\alpha^2-x^2}{x^2}, \qquad x^2 = \frac{\alpha^2}{1+\Omega^2} \tag{85}$$

and

$$K^2 = 2^{2n}(\alpha^2-1). \tag{86}$$

Then substituting this in (83) we may approximate the integrand of Q_A and obtain

$$Q_A \cong \int_0^{\sqrt{\alpha^2-1}} \frac{1}{\Omega^2}\left(\frac{\Omega^2(1+\Omega^2)^n}{\gamma^2K^2\alpha^2}\right)d\Omega = \frac{1}{\gamma^2K^2\alpha^2}\int_0^{\sqrt{\alpha^2-1}}(1+\Omega^2)^n d\Omega. \tag{87}$$

For the gain function of (78a) we use $\gamma^2=\epsilon^2$; for the gain function of (78b) we use $\gamma^2=2\epsilon^2$.

Finally we may write $C(\epsilon^2 \ll 1)$ from (62a) and (62b) as

$$C = \frac{\epsilon^2 f(\alpha^2)}{\alpha^2\pi^2} \tag{88}$$

with $f=f_1$ or f_2 depending on whether the gain function we employ is that of (80a) or (80b). In either case we may use the approximation (33) which applies when $1<\chi<\alpha$, and estimate $f_1(\alpha^2)$, $f_2(\alpha^2)$ with the aid of (57b) and (57d) as

$$\begin{aligned} f_1(\alpha^2) &= (\alpha^2-x^2)\cos(2n\phi+\delta)\big|_{x=\alpha} \\ &= 2(\alpha^2-x^2)\cos^2(n\phi+\xi) - (\alpha^2-x^2)\big|_{x=\alpha} \\ &= \tfrac{1}{2}(2\alpha)^{2(n+1)}(\alpha^2-1). \end{aligned} \tag{89a}$$

Also

$$f_2(\alpha^2) = \tfrac{1}{2}f_1(\alpha^2). \tag{89b}$$

A final, important result concerning the range of prescribed load resistance can be deduced for the equal-ripple gain functions of (78a) and (78b). Note by (26) and (82c) that the numerator polynomial $|s_{11}|^2$ as a function of x is of degree $2(n+1)$. Hence the degree of the numerator polynomial of $s_{11}(\lambda)$ cannot exceed $n+1$. Also, the zeros of s_{11} occur as the argument of $\cos(n\phi+\xi)$ ranges over the interval $0\leq(n\phi+\xi)\leq(n+1)\pi$, so that there are $(n+1)$ real zeros in s_{11}, i.e., all its zeros are at real frequencies. Furthermore when $x=0$, $\xi=\pi/2$ [(8a)], and if n is even then $(n\phi+\xi)|_{x=0}$ is an odd multiple of $\pi/2$, so that $\cos(n\phi+\xi)$ has a zero at $x=0$ when n is even corresponding to a zero at $\lambda=\infty$. We may summarize as follows: All the zeros of $s_{11}(\lambda)$ occur on the $j\Omega$ axis (real frequencies), and the degree of the numerator is n when n is even (to allow for the zero at $\lambda=\infty$), and $n+1$ when n is odd. Thus the equiripple gain functions discussed here result in functions $s_{11}(\lambda)$, *which have no right- or left-half plane zeros* (they are all on the boundary). The resultant networks must then have $s_{11}(\lambda) = s_{22}(\lambda)$, and $P_1=P_2=0$. It then follows from the table that $CQ^2=r_{L1}=r_{L2}$, $1/CQ^2=r_{L3}=r_{L4}$. Further, (74) requires $r_{L1}\geq r_{L3}$, hence $CQ^2\geq 1/CQ^2$; therefore $CQ^2\geq 1$. *Therefore the equiripple gain functions of* (78a) *and* (78b) *result in terminating loads which are always adjustable* between $(1/CQ^2)\leq r_L\leq CQ^2\geq 1$.

A major result of this paper may therefore be summarized as the following theorem:

Theorem: The equiripple band-pass gain function of (79) containing a zero of transmission (infinite loss) at dc may always be realized as a cascaded transmission line structure with, at most, two short-circuiting stubs. Using this function, a load to generator resistance ratio $r_L=R_L/R_g$ may be realized anywhere between the limits

$$\frac{1}{r_{L1}} \leq r_L \leq r_{L1}$$

where, [in (82), (83), (88), and (89)]

$$r_{L1} = CQ^2 \geq 1.$$

Thus, a load resistance equal to the generator resistance is *always* realizable by appropriate stub extraction from the equal ripple gain functions considered here.

As an example let us consider the synthesis of a two-line cascade with a single shunting stub

$$|s_{12}|^2 = \frac{1-\epsilon^2}{1+\epsilon^2\cos(2n\phi+\delta)}.$$

Choose the following parameters

$$\epsilon^2 = 0.01$$
$$n = 2$$
$$\alpha = 2.$$

Let us determine the approximate value of load resistance $R_0=CQ^2$ associated with a design which places the stub at the head end of the chain. Since, by our theorem, this load resistance must equal or exceed unity, we may design a structure by which appropriate stub extraction realizes any termination between R_0 and $1/R_0$.

We may use the approximating formulas to evaluate R_0. Thus, by (87), since we are using the gain function of (78b), [see (86) for K^2]

$$\begin{aligned} Q_A &= \frac{1}{2\epsilon^2K^2\alpha^4}\int_0^{\sqrt{\alpha^2-1}}(1+\Omega^2)^n d\Omega \\ &= \frac{1}{2(0.01)(48)16}\left[\Omega+\frac{\Omega^3}{3}+\frac{\Omega^5}{5}\right]_0^{\sqrt{3}} \\ &= 0.54. \end{aligned}$$

We evaluate Q_B by (80b)

$$Q_B = \frac{1}{\sqrt{\alpha^2 - 1}} \ln \frac{1}{\epsilon^2} = \frac{\ln 100}{\sqrt{3}} = 2.66.$$

The value of $f_1(\alpha^2)$ is approximated by (89a)

$$f_1(4) = \tfrac{1}{2}(4)^6(3) = 2(48)(64).$$

Thus, according to (88)

$$C = \frac{0.01(2)(48)(64)}{4\pi^2} = 1.55.$$

Hence, the terminating resistance is estimated at

$$R_L = C(Q_A + Q_B)^2$$
$$\cong 1.55(3.20)^2 = 15.9\Omega.$$

By (61) the stub characteristic impedance is approximated as

$$Z_S = \frac{1}{Y_S} = \frac{R_g Q}{2\pi} \cong \frac{3.20}{2\pi} = 0.513.$$

If the gain function is constructed according to (7) we have

$$\cos\left[2(2\phi) + \delta\right] = \frac{1}{4 - x^2} \cdot \left[x^6(56+32\sqrt{3}) + x^4(88+48\sqrt{3}) + x^2(39+16\sqrt{3}) - 4\right].$$

[Note that this gives $f_1(\alpha^2) = 2(48)^2$ rather than the approximated value 2(48)(64).]

The function

$$|s_{12}|^2 = \frac{1 - \epsilon^2}{1 + \epsilon^2 \cos(2n\phi + \delta)}$$

for $\epsilon^2 = 0.01$, $n = 2$ is plotted on Fig. 9.

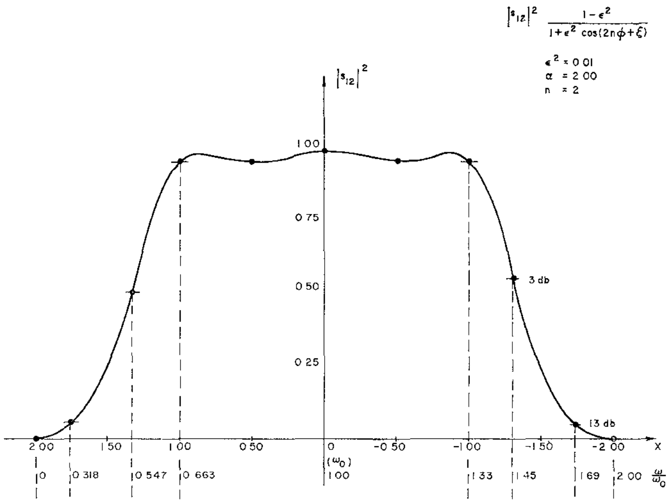


Fig. 9. Band-pass response for cascaded line filter with a single stub and two lines.

The input reflection factor without right-half plane poles is now obtained as

$$s_{11}(\lambda) = -\frac{0.825\lambda^2 + 3.46}{\lambda^3 + 3.355\lambda^2 + 4.305\lambda + 3.46}.$$

Note that, as expected, the zeros of $s_{11}(\lambda)$ are on $j\Omega$, and the numerator degree is $n = 2$, since n is even. This causes no difficulty in the integral of (58), since the boundary singularities of the integrand are logarithmic and the contributions of the small semicircles which avoid these singularities are zero. Thus the principal value of the integral may be used.

The input admittance is now determined from $s_{11}(\lambda)$ as

$$Y(\lambda) = \frac{\lambda^3 + 4.18\lambda^2 + 4.305\lambda + 6.92}{\lambda^3 + 2.53\lambda^2 + 4.305\lambda}.$$

Thus, if the stub is immediately extracted

$$Y_S = \text{res } Y(\lambda)\big|_{\lambda=0} = 1.605$$

$$Z_S = \frac{1}{1.605} = 0.623$$

as compared with the approximate 0.513.

If the cascade lines are now synthesized, the result is a 1-to-17.5 ratio transformer as shown in Fig. 10. The terminating resistor by exact computation is 17.5 ohms as compared with the approximate 15.9 ohms. It should be noted that this is perhaps better agreement than might normally be expected considering the nature of the approximations. In any case, we are assured that $R_L = 1$ is realizable and indeed any load resistance between $(1/17.5) \leq R_L \leq 17.5$ may be achieved with the prescribed gain function. It is rather remarkable that a wide variety of broadband impedance transformers are therefore realizable from the same insertion gain function; e.g., in this case all impedance ratios between 1-to-17.5 and 1-to-$\frac{1}{17.5}$ are realizable with two lines and, at most, two stubs, all transformers having the same reflection losses over the band.

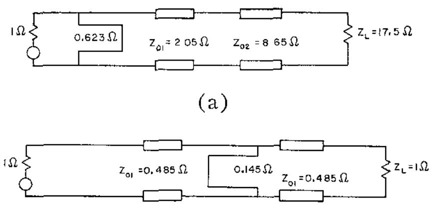


Fig. 10. Alternate designs of band-pass transmission line structure. (a) 17.5-to-1 band-pass transformer. (b) Band-pass filter with one-ohm terminations.

If $R_L = 1$ is desired the stub extraction technique may be applied and the result is shown in Fig. 10. In this case only one stub, rather than the generally required two, was needed, located midway down the chain, as indicated in the figure.

References

[1] Carlin, H. J., Cascade transmission line synthesis, Rept PIBMRI-889-61, Microwave Research Inst., Polytechnic Institute of Brooklyn, N. Y., Apr 1961.
[2] Richards, P. I., Resistor-transmission-line circuits, *Proc. IRE*, vol 37, Feb 1948, pp 217–220.
[3] Collin, R. E., Theory and design of wide-band multisection quarter-wave transformers, *Proc. IRE*, vol 43, Feb 1955, pp 179–185.
[4] Kasahara, Y., H. Ozaki, and T. Fujisawa, Design of distributed constant filter, Osaka Univ. Tech. Rept., vol 4, no 115, 1954.
[5] Ozaki, H., and J. Ishii, Synthesis of transmission-line networks and the designs of UHF filters, *IRE Trans. on Circuit Theory*, vol CT-2, Dec 1955, pp 325–336.
[6] Ikeno, N., The design theory of networks with distributed constants, *Rept. of Res. and Dev.*, (Japan Tel. and Tel. Co.), vol 4, Jul 1955, pp 379–417.
[7] ——, Design and distributed constant networks, *Elec. Commun. Lab., Tech. J.*, vol 4, May 1956, pp 29–36.
[8] Jones, E. M. T., Synthesis of wide-band microwave filters to this prescribed insertion loss, *1956 IRE Conv. Rec.*, pt V, pp 119–126.
[9] Riblett, H. J., General synthesis of quater-wave impedance transformers, *IRE Trans. on Microwave Theory and Techniques*, vol MTT-5, Jan 1957, pp 36–43.
[10] Young, L., Stepped-impedance transformers and filter prototypes, *IRE Trans. on Microwave Theory and Techniques*, vol MTT-10, Sep 1962, pp 339–359.
[11] Wenzel, R. J., Exact design of TEM microwave networks using quater-wave lines, *IRE Trans. on Microwave Theory and Techniques*, vol MTT-12, Jan 1964, pp 94–111.
[12] Sharpe, C. B., A general Tchebycheff, rational function, *Proc. IRE*, vol 42, Feb 1954, pp 454–457.
[13] Helman, D., Tchebycheff approximations for amplitude and delay with rational functions, *Proc. of Symp. on Modern Network Synthesis*, Polytechnic Institute of Brooklyn, N. Y., vol 5, Apr 1955, pp. 385–402.
[14] Carlin, H. J., and W. Kohler, Network theory with transmission line elements, Rept PIBMRI-1046-62, Microwave Research Inst., Polytechnic Institute of Brooklyn, N. Y., Jun 25, 1962.
[15] The functions discussed in the present paper and in [14] are presented from a somewhat different point of view in Riblett, H. J., The application of a new class of equal-ripple functions to some familiar transmission-line problems, *IEEE Trans. on Microwave Theory and Techniques*, vol MTT-12, Jul 1964, pp 415–421.
[16] Carlin, H. J., and A. B. Giordano, *Network Theory*. Englewood Cliffs, N. J.: Prentice-Hall, Jun 1964, ch 4.
[17] Bode, H. W., *Network Analysis and Feedback Amplifier Design*. Princeton, N. J.: Van Nostrand, 1945, pp 363–367.
[18] Fano, R., Theoretical limitations on broadband matching of arbitrary impedances, *J. Franklin Inst.*, vol 249, Jan–Feb 1950, pp 57–98, and pp 139–154, (see Tables I, II, and III, pp 68–69).

Transverse Magnetic Wave Propagation in Sinusoidally Stratified Dielectric Media

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***Abstract*—The problem of the propagation of TM waves in a sinusoidally stratified dielectric medium is considered. The propagation characteristics are determined from the stability diagram of the resultant Hill's equation. Numerical results show that the stability diagrams for Hill's equation and those for Mathieu's equation are quite different. Consequently, the dispersion properties of TM waves and TE waves in this stratified medium are also different. Detailed dispersion characteristics of TM waves in an infinite stratified medium and in waveguides filled longitudinally with this stratified material are obtained.**

Introduction

THE PROBLEM OF electromagnetic wave propagation in a sinusoidally varying dielectric medium is not only of interest from a theoretical point of view but also possesses many possible applications [1], [2]. For example, a section of waveguide filled with this type of inhomogeneous dielectric may be used as a bandpass filter in the mm or in the optical range. The use of an ultrasonic standing wave as a modulating device for certain pressure sensitive media, such as carbon disulfide, pentane, or nitric acid at optical frequencies to achieve a sinusoidally varying dielectric medium may be proposed. Other applications, such as the study of acoustically modulated plasma column and the analysis of sinusoidally modulated dielectric slab antenna, have also been proposed. Furthermore, the results should be very useful in the study of wave propagation in solids [3].

It can be shown [2] that two types of waves, propagating in the direction of the dielectric inhomogeneity, may exist: one with its electric vector transverse to the direction of propagation called a TE wave, and the other with its magnetic vector transverse to the direction of propagation, called a TM wave. The resultant differential equations for TE waves and TM waves are, respectively, the Mathieu and the Hill differential equations [4], [5]. The simpler case of the propagation of TE waves in a sinusoidally stratified dielectric medium has been considered most recently by Tamir, Wang and Oliner [1], and discussed briefly by Yeh and Kaprielian [2]. The purpose of this paper is to consider the problem of the propagation of TM waves in such an inhomogeneous medium. Since the solution of a Hill equation is required, it is expected that the results will be rather in-

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